UNIT-II
VECTOR CALCULUS
PART-A

1. Is the vector $x\vec{i} + 2y\vec{j} + 3z\vec{k}$, Irrotational? **(AU-2009)**

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = \vec{i}\,(0-0) - \vec{j}\,(0) + \vec{k}\,(0-0) = 0$$

$\therefore\ \vec{F}$ is irrotational.

2. Find the divcurl $\vec{F} = x^2 y\vec{i} + xz\vec{j} + 2yz\vec{k}$ **(AU-2010)**

$$\textbf{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xz & 2yz \end{vmatrix} = \vec{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 y)\right] + \vec{k}\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(x^2 y)\right]$$

$$= \vec{i}(2z - x) - \vec{j}(0) + \vec{k}(z - x^2)$$

Divcurl $\vec{F}$ = $\nabla.$**curl** $\vec{F}$

$$= \nabla.\left(\vec{i}(2z-x) - \vec{j}(0) + \vec{k}(z-x^2)\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(\vec{i}(2z-x) - \vec{j}(0) + \vec{k}(z-x^2)\right)$$

$$= \left(\frac{\partial}{\partial x}(2z-x) + \frac{\partial}{\partial z}(z-x^2)\right).$$

$$= -1 + 1 = 0$$

3. If $\nabla \vec{F}$ =yz$\vec{i}$ +xz$\vec{j}$ +xy$\vec{k}$ then find F **(AU-2010)**

$$\nabla \vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$$\vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$$

Equating the coefficient of $\vec{i}$, $\vec{j}$, $\vec{k}$

$$\frac{\partial f}{\partial x} = yz,\quad \frac{\partial f}{\partial y} = xz,\quad \frac{\partial f}{\partial z} = xy$$

$\int \partial f = \int yz\partial x$ $\int \partial f = \int xz\partial y$ $\int \partial f = \int xy\partial z$

f_1=xyz+f(y,z) f_2=xyz+f(x,z) f_1=xyz+f(x,y)

F=xyz+c

4. Find the unit normal to the surface $x^2 + y^2 - 2z + 3 = 0$ at (1, 2,-1) **(AU-2011)**

Given **Φ =** $x^2 + y^2 - 2z + 3 = 0$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right)\left(x^2 + y^2 - 2z + 3\right)$$

$$= 2x\vec{i} + 2y\vec{j} - 2\vec{k}$$

$$(\nabla\phi)_{1,2,-1)} = \left(2\vec{i} + 4\vec{j} - 2\vec{k}\right), |\nabla\phi| = 2\sqrt{6}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 4\vec{j} - 2\vec{k}}{2\sqrt{6}}$$

5. In what direction from (3,1,-2) is the directional derivative of $\phi = x^2 y^2 z^4$ maximum? Find Also the magnitude of this maximum. **(AU-2015)**

$\nabla\phi = 2xy^2z^4\vec{i} + 2x^2yz^4\vec{j} + 4x^2y^2z^3\vec{k}$

At(3,1,-2), $\nabla\phi = 96\left(\vec{i} + 3\vec{j} - 3\vec{k}\right)$

Direction of Maximum $= \nabla\phi = 96\left(\vec{i} + 3\vec{j} - 3\vec{k}\right)$

Magnitude= $|\nabla\phi| = 96\sqrt{1+9+9} = 96\sqrt{19}$.

6. Prove that $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ is irrotational. **(AU-2012)**

$$\nabla X\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ yz & zx & xy \end{vmatrix}$$

$\vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z) = \vec{0}$

$\therefore$ $\vec{F}$ is irrotational.

7. Find λ so that $\vec{F} = (3x-2y+z)\vec{i} + (4x+\lambda y - z)\vec{j} + (x-y+2k)\vec{k}$ is solenoidal

(AU-2015)-2

Given $\vec{F}$ is solenoidal then $\nabla\cdot\vec{F}$ =0

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial y}\vec{k}\right)\cdot\left((3x-2y+z)\vec{i} + (4x+\lambda y - z)\vec{j} + (x-y+2k)\vec{k}\right)$$

$3 + \lambda + 2 = 0$

$\lambda = -5$

8. If $\vec{A}$ *and* $\vec{B}$ are irrotational , prove that $\vec{A}\times\vec{B}$ is solenoidal. **(AU-2013)**

If $\vec{A}$ *and* $\vec{B}$ are irrotational.

$\nabla\times\vec{A} = 0, \nabla\times\vec{B} = 0$

We know that $\nabla.(\vec{A}\times\vec{B}) = (\nabla\times\vec{A}).\vec{B} - (\nabla\times\vec{B}).\vec{A} = 0 - 0 = 0$

$\therefore \vec{A}\times\vec{B}$ is solenoidal.

9. Define solenoidal vector function. If $\overline{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$ is solenoidal, then find the value of λ **(AU-2013)**

Given that $\overline{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k}$

$\nabla.\overline{V} = 0$ if $\overline{V}$ is solenoidal

$$\nabla.\overline{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(x+3y)\vec{i} + (y-2z)\vec{j} + (x+2\lambda z)\vec{k} = 0$$

$$= \left(\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+2\lambda z)\right) = 0$$

= 1+1+2 λ =0

λ =-1

10. Find the value of the constant a, b, c so that the vector $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational **(AU-2010)**

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2)$$

Given $\vec{F}$ is irrotational , $\nabla \times \vec{F} = 0$.

$$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$$

Each component should be zero.

C+1=0, a-4 =0, b-2 = 0

C=-1, a=4, b=2.

11. Prove that $\nabla . r^n = nr^{n-2} . \vec{r}$ **(AU-2011)**

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad |\vec{r}| = \sqrt{x^2+y^2+z^2}$ **,**

$$\nabla . r^n = \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\left[nr^{n-1}\frac{\partial r}{\partial x}\right] + \vec{j}\left[nr^{n-1}\frac{\partial r}{\partial y}\right] + \vec{k}\left[nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= nr^{n-1}\left[\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial x} + \vec{k}\frac{\partial r}{\partial x}\right]$$

$$= nr^{n-1}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= nr^{n-1}\frac{1}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= nr^{n-2}\,\vec{r}$$

12. Find the directional derivative $\phi = x^2 + y^2 + z^2$ in the direction of the vector

$\vec{F} = \vec{i} + 2\vec{j} + 2\vec{k}$ at (1,1,1) **(AU-2014)**

Unit normal vector $\hat{n}$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$ **is** $\left(\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}\right)$

Directional derivative $= \nabla\phi \cdot \hat{n}$

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$grad\,\phi = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$$

$$\nabla\phi_{at(1,1,1)} = 2\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}$$

Directional derivative = $\nabla\phi \cdot \hat{n} = \left(2\,\vec{i} + 2\vec{j} + 2\,\vec{k}\right) \cdot \left(\frac{\vec{i} + 2\,\vec{j} + 2\,\vec{k}}{3}\right) = \frac{10}{3}$

13. Find the unit normal vector to the surface $x^2+y^2=z$ at (1,-2,5) **(AU-2014)**

$\phi =$ $\mathbf{x^2+y^2-z}$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial \vec{z}}\vec{k}\right)\ \mathbf{x^2+y^2-z}$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$(\nabla\phi)_{1,-2,5)} = (2\vec{i} - 4\vec{j} - \vec{k}), |\nabla\phi| = \sqrt{21}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{21}}$$

14. Show that $\vec{F} = (x^2\vec{i} + y^2\vec{j} + z^2)\vec{k}$ is a conservative vector field. **(AU-2009)**

If $\hat{F}$ is conservative then $\nabla \times \hat{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$

Therefore $\hat{F}$ is a conservative vector field.

15. Prove that curl(grad ϕ)=0 **(AU-2014)**

curl(grad ϕ)= curl($\nabla\phi$)=$\nabla X\ \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) X \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial \vec{z}}\vec{k}\right)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \vec{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] - \vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \vec{k}\left[\frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial^2\phi}{\partial y\partial x}\right] = 0$$

16. Find $Curl\vec{F}\ if\ \vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ **(AU-2014)**

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}(-y) - z\vec{j} - x\vec{k}$$

17. If $\vec{F} = x^2\vec{i} + xy\vec{j}$ evaluate $\int_c \vec{F}.d\vec{r}$ from (0,0) to (1,1) along the line y = x. **(AU-2010)**

Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

Along the line y = x, dy = dx

$\therefore \vec{F} = x^2\vec{i} + x.x\vec{j}, \quad d\vec{r} = dx\vec{i} + dyj = dx\vec{i} + dx\vec{j}$

$\vec{F}.d\vec{r} = (x^2\vec{i} + x^2\vec{j}).(dx\vec{i} + dy\vec{j})$

$= x^2dx + x^2dx = 2x^2dx$

$$\int_c \vec{F}.d\vec{r} = \int_0^1 2x^2dx = \frac{2}{3}$$

18. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_c \vec{F}.dr$ Where C is the part of the curve $y = x^2$ between $x = 1$ and $x = 2$. **(AU-2012)**

$\vec{F}.d\vec{r} = (5xy\vec{i} + 2y\vec{j}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$

= 5xydx+2ydy

The curve C: y = x^2

dy=2xdx

x varies from 1 to 2

$$\int_c \vec{F}.d\vec{r} = \int_1^2 5x(x^2)dx + 2x^2 2xdx = \left[5\frac{x^4}{4} + \frac{4x^4}{4}\right]_1^2$$

$$= 36 - \frac{9}{4} = \frac{135}{4}$$

19. Find $\iint_s \vec{r}.d\vec{s}$ where S the surface of the tetrahedron whose vertices are is

$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1).$ **(AU-2010)**

By Gauss divergence theorem

$$\iint_S \vec{r}.d\vec{s} = \iiint_V \nabla.\vec{r}dv$$

$$= \iiint_V (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}).(x\vec{i} + y\vec{j} + z\vec{k})dv$$

$$= \iiint_V (1+1+1)dv$$

$$= \iiint_V 3dv$$

$$= 3\int_0^1\int_0^1\int_0^1 dxdydz = 3.$$

20. If S is any closed surface enclosing a volume V and $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k},$ prove that $\iint_S \vec{F}.\hat{n}ds$ =(a+b+c)V. **(AU-2011)**

Gauss Divergence theorem is

$$\iint_S \vec{F}.\hat{n}.ds = \iiint_V \nabla.\vec{F}dV$$

$$= \iiint_V (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}).(ax\vec{i} + by\vec{j} + cz\vec{k})dv$$

$$= \iiint_V (a+b+c)dv$$

= (a+b+c) V

21. State Green's theorem in a plane. **(AU-2010)**

If M(x,y) and N(x,y) and its partial derivatives are continuous and one valued functions in the region R of the xy plane bounded by a simple closed curve C, then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Where C is the curve prescribed in the positive direction.

22. Using Green's theorem, Prove that the area enclosed by a simple closed curve C is $\frac{1}{2}\int (xdy - ydx)$. **(AU-2011)**

By Green's theorem

$$\int_C Mdx + Ndy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Let M = -y N=x

$$\int_C -ydx + xdy = \iint_S (1+1)dxdy$$

$$= 2\iint_S dxdy$$

= 2(area enclosed by C)

Therefore Area enclosed by C = $\frac{1}{2}\int (xdy - ydx)$

23. State Gauss Divergence theorem. **(AU-2012)**

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S then

$$\iint_s \vec{F} \cdot \hat{n}\, ds = \iiint_v \nabla \cdot \vec{F}\, dv$$

24. State Stoke's theorem. **(AU-2015) (2)**

The surface integral of the normal component of the curl of a vector function F over an Open surface S is equal to the line integral of the tangential component of F around the Closed curve C bounding S. $\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}ds$

PART-B

1. a. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Prove that $div(r^n \vec{r}) = (n+3)r^n$ and $curl\,(r^n\, \vec{r}) = 0$.

(AU-2011) (8)

b. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then prove that $div\, grad\,(r^n) = n(n+1)r^{n-2}$.Hence deduce that $divgrad\left(\frac{1}{r}\right) = 0$ **(AU-2015)-2(8)**

2. a. Find the directional derivative of ϕ =3x^2+2y-3z at (1,1,1) in the direction of $2\vec{i}$ +2 $\vec{j}$ - $\vec{k}$ **(AU-2012) (8)**

b. Find the angle between the surfaces x^2+y^2+z^2=9 and z=x^2+y^2-3 at the point (2,-1,2). **(AU-2012)(8)**

3. a. Find the angle between the normal's to the surfaces x^2=yz at the points (1,1,1) and(2,4,1)

(AU-2014)(8)

b. Find the directional derivative of ϕ =x^{2}yz+4xz^2 at P(1,-2,-1) that is maximum and also in the direction of PQ where Q is (3,-3,-2) **(AU-2010 (8)**

4. a. Evaluate $\int_C \phi d\vec{V}$ where C is the curve x=t, y=t^2 , z=1-t and $\phi = x^2 y(1+z)$ from t=0 to t=1 **(AU-2011)(8)**

b. If $\nabla\phi = (x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x-y+2z)\vec{k}$, find the Scalar point function ϕ. **(AU-2011)(8)**

5. a. Find the value of n so that the vector $r^n \vec{r}$ is both solenoidal and irrotational

(AU-2015)-2(8)

` b. Prove that $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ is irrotational and hence find its scalar potential.

(AU-2014)(8)

6. a. Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational .Hence find its scalar potential φ **(AU-2015)(8)**

b. Prove that $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$ is irrotational and hence find its scalar potential. **(AU-2014)(8)**

7. a. Find the work done the force $\vec{F} = 3xy\vec{i} - y^3\vec{j}$ moves a particle along the Curve C: y=2x^{2}from (0, 0) to (1, 2) in the xy-plane. **(AU-2011)(8)**

b. Evaluate $\int_c \vec{f}.d\vec{r}$ where $\vec{f} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ along the straight line Joining (1,- 2,1) and (3, 2, 4) **(AU-2012)(8)**

8. a. Show that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force field. Find the Scalar potential and the work done by $\vec{F}$ in moving an object in this field

from (1,-2, 1) to (3, 1,4) **(AU-2009)(8)**

b. Find the directional derivative of xy^2+yz^3 at (2,-1,1) in the direction of the normal to the surface $x\log z-y^2+4=0$ at (-1,2,1) **(AU-2009)(8)**

9. a .Evaluate $\iint_S \vec{f}.\hat{n}ds$ Where $\vec{f}=(x^2+y^2)\vec{i}-2x\vec{j}+2yz\vec{k}$ and S is the surface of the $2x+y+2z=6$ in the first octant. **(AU-2010)(8)**

b. Using Green's theorem in the plane evaluate $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$ Where C is the boundary of the region enclosed by $y=\sqrt{x}$ *and* $y=x^2$. **(AU-2009)(8)**

10. a. Using Green's theorem ,evaluate $\int_C (y-\sin x)dx+\cos x dy$ where C is the triangle formed by y=0,, $x=\frac{\pi}{2}, y=\frac{2x}{\pi}$ **(AU-2015)(8)**

b. Apply Green's theorem in the plane to evaluate $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$ Where C is the boundary of the region defined by by $x=0,\ y=0$ *and* $x+y=1$. **(AU-2014)-2(8)**

11.a. Verify Green's theorem in a plane for $\int (xy+y^2)dx+x^2dy$ where C is the boundary of the common area between $y = x^2$ and y=x in the xoy plane **(AU-2014)(8)**

b.Using Green's theorem ,evaluate $\int_C (x^2-2xy)dx+(x^2y+3)dy$,where C is the region bounded by the curves $y^2=8x$ and x=2 **(AU-2015)(8)**

12.a. Verify Gauss Divergence theorem for $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ Where S is the surface of the Cuboid formed by the planes $x=0, x=a, y=0, y=b, z=0, z=c.$ **(AU-2014)(8)**

b. Verify Gauss's divergence theorem for the function $\vec{F}=y\vec{i}+x\vec{j}+z^2\vec{k}$ Over the cylindrical region bounded by $x^2+y^2=9,\ z=0$ *and* $z=2.$ **(AU-2012)(8)**

13. Verify Gauss's divergence theorem for $\vec{F}=4xz\vec{i}-y^2\vec{j}+yz\vec{k}$ and C is it's boundary over the cube $x=0, x=1, y=0, y=1, z=0, z=1.$ **(AU-2015)-3(16)**

14. Verify Gauss Divergence theorem for $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ taken over the cube bounded by the Planes x=0,y=0,z=0,x=1,y=1 and z=1 **(AU-2015)(16)**

15.a. Verify stoke's theorem for $\vec{F}=(x^2-y^2)\vec{i}+2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x=0, x=a, y=0, y=b.$ **(AU-2015)-3(8)**

b. Verify Stoke's theorem for the vector field $\vec{F}=(2x-y)\vec{i}-yz^2\vec{j}-y^2z\vec{k}$ where S is the surface of upper hemisphere $x^2+y^2+z^2=1$ and C is its boundary in xy-plane**.** **(AU-2014)(8)**

UNIT-V
LAPLACE TRANSFORMS
PART-A

1. State the sufficient condition for existence of the Laplace transform of f(t) **(AU-2015)**

(i)f(t)should be continuous or piecewise continuous in the given closed interval [a,b] where a>0

(ii) f(t) should be of exponential order.

2. Find the Laplace transform of $f(t) = t\cosh t$ **(AU-2014)**

$$L(t\cosh t) = \frac{-d}{ds}(L(\cosh t)) = \frac{-d}{ds}\left[\frac{s}{s^2-1}\right]$$

$$= -\left[\frac{(s^2-1)(1)-s(2s)}{(s^2-1)^2}\right] = -\left[\frac{-1-s^2}{(s^2-1)^2}\right] = \frac{s^2+1}{(s^2-1)^2}$$

3. Find the Laplace transform of $\frac{t}{e^t}$ **(AU-2013)**

$$\mathbf{L}\left(\frac{t}{e^t}\right) = L(te^{-t}) = L(t)_{s\to s+1} = \left(\frac{1}{s^2}\right)_{s\to s+1} = \frac{1}{(s+1)^2}$$

4. State and prove change of scale property in Laplace transform. **(AU-2012)**

If L(f(t)) = F(s), then L(f(at)) = $\frac{1}{a}F\left(\frac{s}{a}\right)$.

L(f(at)) = $\int_0^\infty e^{-st} f(at)dt$

at=u t=0 u=0

adt=du t=∞ u=∞

L(f(at)) = $\int_0^\infty e^{-\frac{su}{a}} f(u)\frac{du}{a}$

$= \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} f(u)du$

L(f(at)) = $\frac{1}{a}F\left(\frac{s}{a}\right)$

5. State the first shifting theorem on Laplace transforms. **(AU-2012)**

If L(f(t))=F(s) then L[e^{at}f(t)]=F[s-a] and

If L(f(t))=F(s) then L[e-^{at}f(t)]=F[s+a]

6. Find the Laplace transform of $\frac{e^{-2t}}{\sqrt{t}}$ **(AU-2012)**

$$L[t^{-1/2}e^{-2t}] = L[t^{-1/2}]_{s\to s+2}$$

$$= \left[\frac{\sqrt{\pi}}{s}\right]_{s\to s+2}$$

$$= \left[\frac{\sqrt{\pi}}{s+2}\right]$$

7. Find the Laplace transform of $\sqrt{t}e^{3t}$ **(AU-2012)**

$$L[t^{1/2}e^{3t}] = L[t^{1/2}]_{s\to s-3}$$

$$= \left[\frac{\sqrt{\pi}}{2s^{3/2}}\right]_{s\to s-3}$$

$$= \frac{\sqrt{\pi}}{2(s-3)^{3/2}}$$

8.Find $L[\cos^2 3t]$ **(AU-2011)**

$$L[\cos^2 3t] = L\left[\frac{1+\cos 6t}{2}\right]$$

$$= \frac{1}{2}L[1+\cos 6t]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

9. Find $L[(t^3 + 3e^{2t} - 5\sin 3t)e^{-t}]$ **(AU-2011)**

$$L[(t^3 + 3e^{2t} - 5\sin 3t)e^{-t}] = L[(t^3e^{-t} + 3e^{2t}e^{-t} - 5\sin 3te^{-t})]$$

$$= L[(t^3e^{-t} + 3e^{t} - 5\sin 3te^{-t})]$$

$$= L[t^3]_{s\to s+1} + 3L(1)_{s\to s+1} - 5L[\sin 3t]_{s\to s+1}$$

$$= \left[\frac{6}{s^4} + \frac{3}{s} - \frac{3}{s^2+9}\right]_{s\to s+1}$$

$$= \left[\frac{6}{(s+1)^4} + \frac{3}{s+1} - \frac{3}{(s+1)^2+9}\right]_{s\to s+1}$$

10. Find $L\left[\frac{\sin t}{t}\right]$ **(AU-2014)**

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L(\sin t)ds = \int_s^\infty \frac{1}{s^2+1}ds = [\tan^{-1}(s)]_s^\infty = [\tan^{-1}(\infty) - \tan^{-1}(s)]$$

$$= \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s)$$

11. Find the Laplace transform of the function $f(t) = \frac{1-e^{-t}}{t}$ **(AU-2013)**

$$L(f(t) = L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty L(1-e^{-t})ds = \int_s^\infty L(1) - L(e^{-t})ds = \int_s^\infty \left[\frac{1}{s} - \frac{1}{s+1}\right]ds$$

$$= [\log s - \log(s+1)]_s^\infty$$

$$= \log\left[\frac{s+1}{s}\right]$$

12. Verify initial value theorem for f(t)=1+e^{-t}(sint+cost) **(AU-2012)**

Initial value theorem is ,if $L[f(t)]$=F(s),then $\underset{t\to 0}{Lt} f(t) = \underset{t\to\infty}{Lt} sF(s)$

F(s)=L(1+)= $L[1 + e^{-t}(\sin t + \cos t)]$

F(s)= $\frac{1}{s} + L[\sin t + \cos t]_{s\to s+1}$

F(s)= $\frac{1}{s} + \frac{s+2}{(s+1)^2+1}$

$$\underset{t\to 0}{Lt} f(t) = \underset{t\to 0}{Lt}\ 1 + e^{-t}(\sin t + \cos t) \quad = 2$$

$$\underset{t\to\infty}{Lt}\, sF(s) = \underset{s\to\infty}{Lt}\, s\left[\frac{1}{s} + \frac{s+2}{(s+1^2)+1}\right] \quad = \underset{s\to\infty}{Lt}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right] = 2$$

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s) = 2$$

Hence the initial value theorem is verified**.**

13. Verify Initial value theorem for the function f(t) = ae^{-bt} **(AU-2013)**

$$f(t) = ae^{-bt}\text{, } F(s) = L[f(t)] = L[ae^{-bt}] = \frac{a}{s+b}$$

Initial value theorem: $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$

L.H.S = $\underset{t\to 0}{Lt}\, f(t) = \underset{t\to 0}{Lt}\, ae^{-bt} = a$

R.H.S $\underset{s\to\infty}{Lt}\, sF(s) = \underset{s\to\infty}{Lt}\, s\left(\frac{a}{s+b}\right) = \underset{s\to\infty}{Lt}\frac{as}{s+b}$

$$= \underset{s\to\infty}{Lt}\frac{as}{\left(1+\frac{b}{s}\right)} = \underset{s\to\infty}{Lt}\frac{a}{1+\left(\frac{b}{s}\right)} = a$$

Hence the initial value theorem is verified**.**

14. If $L\left(e^{-t}\cos^2 t\right) = F(s)$, *find* $\lim_{s\to 0} sF(s)$ **(AU-2013)**

F(s)=L(e^{-t}cos^{2}t)= $L\left[\cos^2 t\right]_{s\to s+1}$

$$= L\left[\frac{1+\cos 2t}{2}\right]_{s\to s+1}$$

$$= \frac{1}{2}L\left[1+\cos 2t\right]_{s\to s+1} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]_{s\to s+1}$$

L(e^{-t}cos^{2}t)= $\frac{1}{2}\left[\frac{1}{s+1} + \frac{s}{(s+1)^2+4}\right]$

15. Define periodic function with an example. **(AU-2010)**

A function f(t) is said to have a period T or to be periodic with period T if for all t, f(t+T) = f(t) where T is a positive constant. The least value of T>0 is called the period of f(t) .

$$f(t) = \sin t$$

$$f(t+2\pi) = \sin(t+2\pi)$$

Eg. Consider $\qquad = \sin t$

$$i.e. f(t) = f(t+2\pi) = \sin t$$

$$\therefore \sin t \text{ is a periodic function with period } 2\pi$$

16. Evaluate $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$ **(AU-2014)**

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right] = L^{-1}\left[\frac{1}{s^2+6s+9+4}\right] = L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right] = e^{-3t}\frac{\sin 2t}{2}$$

17. Find the Laplace inverse transform of $\frac{1}{(s+1)(s+2)}$ **(AU-2012)**

$$\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$$

A=1 and B=1(using partial fraction)

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=L^{-1}\left[\frac{1}{s+1}\right]+L^{-1}\left[\frac{1}{s+2}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=e^{-t}+e^{-2t}$$

18. Find the inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$ **(AU-2012)**

We know that $L^{-1}[F(s)]=\frac{-1}{t}L^{-1}[F'(s)]$

$$L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]=\frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]\right]$$

$$=\frac{-1}{t}L^{-1}\left[\frac{d}{ds}[\log(s+1)-\log(s-1)]\right]$$

$$=\frac{-1}{t}L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right]=\frac{-1}{t}\left[e^{-t}-e^{t}\right]=\frac{2}{t}\sinh t$$

19. Find the Laplace transform of $\int_0^t te^{-t}dt$ **(AU-2015)**

$$L\left[\int_0^t te^{-t}dt\right]=\frac{1}{s}L[te^{-t}]$$

$$=\frac{1}{s}\left[-\frac{d}{ds}L[e^{-t}]\right]$$

$$=\frac{1}{s}\left[-\frac{d}{ds}\left(\frac{1}{s+1}\right)\right]$$

$$=\frac{-1}{s}\left(\frac{1}{(s+1)^2}\right)=\frac{-1}{s(s+1)^2}$$

20. Find the inverse Laplace transform of $\frac{e^{-\pi s}}{(s-1)^2}$ **(AU-2014)**

$$L^{-1}\left(\frac{1}{s^2}\right)=t \quad and \quad L^{-1}\left(\frac{1}{(s-1)^2}\right)=te^{t}$$

$$L\left(\frac{e^{-\pi s}}{(s-1)^2}\right)=(t-\pi)e^{(t-\pi)}$$

PART-B

1. a .Find $L\left(t^2e^{-3t}\sin 2t\right)$ **(AU-2013)(8)**

   b. Find $L\left[\frac{\cos at-\cos bt}{t}\right]$ **(AU-2015)(8)**

2. a. Find the Laplace transform of the square-wave function (or Meoander function) of

   Period a defined as $f(t)=\begin{cases}1, & when 0<t<\frac{a}{2}\\ -1, & when \frac{a}{2}<t<a\end{cases}$ **(AU-2013)(8)**

   b. Find the Laplace transform of the following triangular wave function given by

$f(t)=\begin{cases} t & 0\le t\le c \\ 2c-t & c\le t\le 2c \end{cases}$ and $f(t+2c)=f(t)$. **(AU-2015)(8)**

3. a. Find the Laplace transform of the periodic function defined on the interval by $0\le t\le 1$

$f(t)=\begin{cases} -1, & 0\le t\le \frac{1}{2} \\ 1 & \frac{1}{2}\le t<1 \end{cases}$ and $f(t+1)=f(t)$. **(AU-2014)(8)**

b. Find the Laplace transform of $f(t)=t^2$, $0<t<2$ and $f(t+2)=f(t)$ for all $t>0$ **(AU-2013)(8)**

4. a. Find the inverse Laplace transform of $\frac{4s+7}{s^2(2s+3)(3s+5)}$ **(AU-2013)(8)**

b. Find $L^{-1}(s/(s^2+1)(s^2+4))$ **(AU-2015)(8)**

5.a. Find the Laplace transforms of the following functions 1)$\dot{e}^t t\cos t$ 2)1-cost/t π/w **(AU-2014)(8)**

b. Find the Laplace transform of $f(t)= te^{-3t}\cos 2t$. **(AU-2014)(8)**

6. a. Find the Laplace transforms of $f(t)=\begin{cases} t & 0\prec t\prec a \\ 2a-t & a\prec t\prec 2a \end{cases}$,

where $f(t+2a)=f(t)$ **(AU-2014)(8)**

b. Find the Laplace transform of f(t) where

$$f(t)=\begin{cases} \sin\omega t & \text{for } 0\prec t\prec \frac{\pi}{\omega} \\ 0 & \text{for } \frac{\pi}{\omega}\prec t\prec \frac{2\pi}{\omega} \end{cases},\ f\left(t+\frac{2\pi}{\omega}\right)=f(t)$$ **(AU-2014)(8)**

7. a. Find the Laplace transform $f(t)=\begin{cases} \cos t & 0\prec t\prec \pi \\ 0 & \pi\prec t\prec 2\pi \end{cases}$, $f(t+2\pi)=f(t)$

b. Find $L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)^2}\right]$ **(AU-2013)(8)**

8.a .Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ and find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ hence find $L^{-1}\left[\frac{1}{(s^2+9s+13)^2}\right]$ **(AU-2013)(8)**

b. Use convolution theorem to find the inverse of $\frac{s}{(s^2+4)(s^2+9)}$ **(AU-2013)(8)**

9. a. Find the Laplace transform of $f(t)=\frac{\cosh t\cos t}{t}$ *and* $g(t)=\sin\sqrt{t}$ **(AU-2013)(8)**

b. Using convolution theorem to find the inverse Laplace transform of the function

$\frac{s}{(s^2+1)^2}$ **(AU-2014)(8)**

10.a. Using convolution theorem to find the inverse Laplace transform of the function

$\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ **(AU-2014)-2(8)**

b .Using convolution, solve the initial value problem,y”+9y=sin3t, y(0)=0,y’(0)=0.

11. a. Verify initial and final value theorem for $f(t)=1+e^{-t}(\sin t+\cos t)$ **(AU-2014)(8)**

b. Verify initial and final value theorem for $f(t)=1+e^{-2t}$

12.a. Solve $y''+y'=t^2+2t$, $y(0)=4$, $y'(0)=-2$ by using Laplace transform. **(AU-2013)(8)**

b. Solve the differential equation $y''-3y'+2y=4t+e^{3t}$ where y(0)=1 and y^1 **(0)=-1** using Laplace transforms **(AU-2015)(8)**

13.a Solve $y''-6y'+9y=t^2e^{3t}, y(0)=2, y'(0)=6$ by Laplace transform method **(AU-2014)(8)**

b. Solve the following differential equation ,using Laplace transform

$y''+2y'+2y=8e^t\sin t$, $y(0)=y'(0)=0$ **(AU-2013)(8)**

14.a. Using Laplace Transform , solve $\frac{d^2y}{dt^2}+4y=\sin 2t$ given y(0) =3, y'(o)=4 **(AU-2014)(8)**

b.Use Laplace transform to solve $(D^2-3D+2)y=e^{3t}$ with y(0)=1 ,$y^1(0)=0$ **(AU-2014)(8)**

15.a. Using Laplace transform method, solve $d^2y/dt^2+9y=18t$ given that y(0)=0,y(π/2)=0.

b. **U**sing Laplace transform ,find the solution of $y'+3y+2\int_0^t y\,dt=t$ $y(0)=0$

UNIT-III
ANALYTIC FUNCTIONS
PART- A

1. State the basic difference between the limit of a function of a real variable and that of a complex variable. (AU2012)

Real variable	Complex Variable
Limit takes along x axis and y axis or parallel to both axis	Limit takes along any path (straight or curved)

2. State the necessary condition of Cauchy-Riemann equations **(AU-2011)**

The necessary condition for $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

3. Write 2-D Laplace equations in polar coordinates. **(AU-2011)**

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

4. Show that the function $f(z) = \bar{z}$ is nowhere differentiable. **(AU-2014)-2**

Given $f(z) = \bar{z} = x - iy$

$u = x, v = -y$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial y} = -1$$

$u_x \neq v_y$, C-R equations are not satisfied anywhere. Hence

$f(z) = \bar{z}$ is nowhere differentiable.

5. Find the constants a,b if $f(z) = x + 2ay + i(3x + by)$ is analytic **(AU-2013)**

$f(x) = x + 2ay + i(3x + by)$

$u = x + 2ay \quad and \quad v = (3x + by)$

Where $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 2a$

$$\frac{\partial v}{\partial x} = 3, \quad \frac{\partial v}{\partial y} = b$$

We know that by CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$a = \frac{3}{2}, \quad b = 1$$

6. If u+iv is analytic , show that v –iu &-v +iu are also analytic **(AU-2013)**

Given u+iv is analytic

C –R equations are satisfied $u_x = v_y$ …….. (1)

$u_y = -v_x$ ……..(2)

Since the derivatives of u & v exis it is therefore continuous

Now to prove v –iu & -v+iu are also analytic, we should prove that

(i) $v_x = -u_y$ & $v_y = u_x$ &

(ii) $v_x = u_v$ & $v_y = u_x$

(iii) u_x, u_y, v_x, v_y are all continuous . Results (i) & (ii follows from (1) & (2) . since the derivatives of u &v exists from (1) and (2) , the derivatives of u and v should be continuous .

Hence the result

7. Find the value of a,b,c,d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ **may be analytic** **(AU-2013)**

$$f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$$

$$u = x^2 + axy + by^2 \quad , v = cx^2 + dxy + y^2$$

$$u_x = 2x + ay \quad , v_x = 2cx + dy$$

$$u_y = ax + 2by \quad , v_y = dx + 2y$$

$$f(z) \text{ is analytic}, \; u_x = v_y \quad and \;\; u_y = -v_x$$

$$a = 2, b = -1, c = -1, d = 2$$

8. State whether or not $f(z) = \bar{z}$ **is an analytic function** **(AU-2012)-2**

$$w = f(z) = \bar{z}$$

$$u + iv = x - iy \Rightarrow u = x \text{ and } v = y$$

$$u_x = 1, \quad v_x = 0$$

$$u_y = 0, \quad v_y = -1$$

$$u_x \neq v_y$$

Hence CR equations are not satisfied

$\therefore$ The function $f(z)$ is nowhere analytic

9. Verify whether or not $f(z) = e^x(\cos y - i\sin y$ **is analytic** **(AU-2014)**

$$u = e^x \cos y \quad and \quad v = -e^x(\sin y)$$

$$\frac{\partial u}{\partial x} = e^x \cos y \;\; and \;\; \frac{\partial v}{\partial x} = -e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \;\; and \;\; \frac{\partial v}{\partial y} = -e^x \cos y$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} and \;\; \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

CR equations are not satisfied. It is not an analytic function.

10. S.T $f(z) = e^x \sin y$ is **harmonic** **(AU-2014)**

$$u_x = e^x \sin y, \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = o$$

$f(z) = e^x \sin y$ is **harmonic**

11. If $f(z)$ **is an analytic function whose real part is constant, Prove that** $f(z)$ **is a constant function.** **(AU-2012)**

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function

Therefore by CR equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

Given $u=$constant

To prove $f(z)$ is a constant

$u=c$

$\frac{\partial u}{\partial x}=0, \quad \frac{\partial u}{\partial y}=0$

By CR equation $\frac{\partial u}{\partial x}=0 \Rightarrow \frac{\partial v}{\partial y}=0 \;\; and \quad \frac{\partial u}{\partial y}=0 \Rightarrow \frac{\partial v}{\partial x}=0$

$\therefore f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0+i0$

$f'(z)=0 \Rightarrow f(z)=c$

$f(z)$ is a constant.

12. Find the image of the line x=k under the transformation $\omega=\frac{1}{z}$ (AU-2013)

Given $w=\frac{1}{z}$

i.e., $z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$

$x+iy=\frac{u}{u^2+v^2}+i\left(\frac{-v}{u^2+v^2}\right)$

i.e., $x=\frac{u}{u^2+v^2}$............(1), $y=\frac{-v}{u^2+v^2}$............(2)

Given x=k in the z plane

$k=\frac{u}{u^2+v^2}.by(1),$

$k(u^2+v^2)=u$

$u^2+v^2-\frac{1}{k}u=0$

$\left(u-\frac{1}{2k}\right)^2+v^2-\frac{1}{4k^2}=0$

$\left(u-\frac{1}{2k}\right)^2+v^2=\frac{1}{4k^2}$ which is a circle whose centre is $\left(\frac{1}{2k},0\right)$ and radius $\frac{1}{2k}$

13. Find the map of the circle $|z|$ = 3 under the transformation $w=2z$ (AU-2012)

$w=2z$

$u+iv=2(x+iy)$

$u=2x, v=2y \Rightarrow x=\frac{u}{2}, y=\frac{v}{2}$

Given $|z|=3 \Rightarrow |x+iy|=3 \Rightarrow x^2+y^2=9$

$\therefore \left(\frac{u}{2}\right)^2+\left(\frac{v}{2}\right)^2=9 \Rightarrow u^2+v^2=36$

Hence the image of $|z|=3$ in the z-plane is transformed into

$u^2+v^2=36$ in the w-plane under the transformation $w=2z$

14.Find the image of the circle $|z|$ = 2 under the transformation $\omega=3z$ (AU-2012)

$w=3z$

$u+iv=3(x+iy)$

$u=3x, v=3y \Rightarrow x=\frac{u}{3}, y=\frac{v}{3}$

Given $|z|=2 \Rightarrow |x+iy|=2 \Rightarrow x^2+y^2=4$

$\therefore \left(\frac{u}{3}\right)^2+\left(\frac{v}{3}\right)^2=4 \Rightarrow u^2+v^2=36$

Hence the image of $|z|=2$ in the z-plane is transformed into

$u^2+v^2=36$ in the w-plane under the transformation $w=3z$

15. Find the image of the circle $|z|$ = λ under the transformation $w=5z$ **(AU-2011)**

$w=5z$

$u+iv=5(x+iy)$

$u=5x, v=5y \Rightarrow x=\frac{u}{5}, y=\frac{v}{5}$

Given $|z|=\lambda \Rightarrow |x+iy|=\lambda \Rightarrow x^2+y^2=\lambda^2$

$\therefore \left(\frac{u}{5}\right)^2+\left(\frac{v}{5}\right)^2=\lambda^2 \Rightarrow u^2+v^2=(5\lambda)^2$

Hence the image of $|z|=\lambda$ in the z-plane is transformed into

$u^2+v^2=(5\lambda)^2$ in the w-plane under the transformation $w=5z$

16.Define critical point of a transformation **(AU-2010)**

A point z_0 at which the mapping w=f(z) is not conformal is called the critical point .

17. Find the invariant points of the transformation $f(z)=z^2$ **(AU-2014)**

$f(z)=z^2$

$w=z^2,$

$z=z^2$

$z^2-z=0$

$z(z-1)=0$

$z=0, z=1$

The invariants points are z=0,z=1.

18. Find the critical points of the transformation $w=1+\frac{2}{z}$ **(AU-2013)**

$z=1+\frac{2}{z}$ $\quad z^2-z-2=0$ $\quad (z-2)(z+1)=0$

$z=2, z=-1$

Critical points are z=2,-1

19. Find the invariant points of the transformation $\omega=\frac{2z+6}{z+7}$ **(AU-2013)**

The invariant points are given by $z=\frac{2z+6}{z+7}$

$z^2+7z-2z-6=0 \Rightarrow z^2+5z-6=0$

$(z+6)(z-1)=0$

$z=-6,1$

20. Prove that a bilinear transformation has atmost two fixed points. **(AU-2012)**

The fixed points of the transformation $w=\frac{az+b}{cz+d}$ is obtained from $z=\frac{az+b}{cz+d}$

or $cz^2+(d-a)z-b=0$

These points are two in number unless the discriminant is zero in which case the number of points is one.

21.Show that $|Z|^2$ is not analytic at any point. **(AU-2015)**

f(z)= $|Z|^2 = u+iv=x^2+y^2$ where $u= x^2+y^2$ and $v=0, u_{x=}2x, u_{y=}2y, v_{x=}0, v_y=0$

hence f(z) is not analytic.

PART-B

1. a. If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ **(AU-2013)(8)**

b. show that a harmonic function u satisfies the formal differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$ and hence P.T $\log|f^1(z)|$ is harmonic, where f(z) is a regular function.

(AU-2015)(8)

2. a. Show that the function $u = e^{-x}(x\cos y + y\sin y)$ is harmonic function. Hence find the corresponding analytic function $f(z) == u + iv$ **(AU-2014)(8)**

b. Determine the analytic function $w = u + iv$ given that $u = e^{-2x}(x\cos y + y\sin y)$ **(AU-2015)(8)**

3. a Prove that $u=e^{-y}\cos x$ and $v=e^{-x}\sin y$ satisfy Laplace equations but that u+iv is not an analytic function of z.

b. Find if $\Phi = (x-y)(x^2+4xy+y^2)$ can represent the equipotential surface for an electric field. Find the corresponding complex potential $\omega = \phi + i\psi$ and also ψ **(AU-2013)(8**

4.a. Find the analytic function $f(z) == u + iv$ where $v = 3r^2 \sin 2\theta - 2r\sin\theta$. Verify that u is a harmonic function. **(AU-2013)(8)**

b. Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ **(AU-2014) (8)**

5.a. Prove that the function $u = e^{x}(x\cos y - y\sin y)$ satisfies Laplace's equation and find the corresponding analytic function $f(z) == u + iv$ **(AU-2013)(8)**

b. Prove that the real and imaginary parts of an analytic function are harmonic function**.** **(AU-2014) (8)**

6.a. Find the analytic function $w = u + iv$ if $e^{2x}(x\cos 2y - y\sin 2y)$ and hence find u **(AU-2013)(8)**

b. Find the analytic function $f(z) = u(x, y) + iv(x, y)$ given that $u - v = e^{x}(\cos y - \sin y)$ **(AU-2014)(8)**

7. a . If $u = x^2 - y^2$ and $v = \frac{-y}{x^2+y^2}$ prove that both u and v satisfy Laplace equations, but $u + iv$ is not a regular function of z **(AU-2013)(8)**

b. Find the image of the circle $|z| = 2$ under the transformation $\omega = z + 3 + 2i$ **(AU-2013)(8)**

8.a. Find the image of w plane of the region of the z-plane bounded by the straight line x=1,y=1 and x+y=1 under the transformation $w = z^2$ **(AU-2013)(8)**

b.Find the image in the w-plane of the infinite strip1/4≤y≤1/2under the transformation w=1/z **(AU-2015)(8)**

9.a. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z-plane to the upper half of the w-plane and also find the image of the unit circle of the z plane. **(AU-2013)(8)**

b .Find the image of the circle $|z-3i|=3$ and the region $1 < x < 2$ under the map $w = \frac{1}{z}$

10 .a. Find the image of |z+2i|=2 under the transformation w=1/z.

b. Find the image of the following regions under the transformation w=1/z.

i) the half plane x>c when c>0

ii) the half plane y>c when c<0

11.a. S.T under the mapping w= i-z/i+z ,the image of the circle $x^2+y^2<1$ is the entire half of the w-plane to the right of the imaginary axis

b. Find the image of the region bounded by the lines x=0,y=0,and x+y =1 under the mappings $w = e^{\frac{i\pi}{4}}$ and $w = z + (2+3i)$ **(AU-2014)(8)**

12.a Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation w = $\frac{1}{z}$ is the lemniscates $r^2 = \cos 2\theta$ ***(AU-2012)*(8)**

b. Find the Bilinear transformation which maps z= 0,z=1,z=∞ into the points w=i, w=1,w=-i **(AU-2013)(8)**

13.a. Find the bilinear transformation that maps 1,i,and -1 of the z-plane onto 0,1 and ∞ on the w- plane.Also find the image of the unit circle of the z plane. **(AU-2014) (8)**

b. Find the Bilinear transformation that maps the points z=∞,1,0 onto the points w=o,i, ∞ respectively **(AU-2012)(8)**

14.a. Find the Bilinear transformation that maps the points z=1,i,-1 into the points w=,0,1, ∞ respectively. Find also the pre-image of $|w|$=1 under this bilinear transformation. **(AU-2014)(8)**

b. Find the bilinear transformation that maps the points z=0,-1,i into the points w= i,0,∞ respectively. **(AU-2015)(8)**

15. a. Find the bilinear transformation that maps the points 1+i,-i,2-i of the z- plane into the points 0,1,i of the w-plane.

b. Find the bilinear transformation that maps the points z=i,-1,1 into the points w=0,1,∞ respectively.

UNIT-IV
COMPLEX INTEGRATION
PART-A

1. State Cauchy's integral theorem **(AU-2015)**

If f (z) is analytic inside and on a closed curve c of a simply connected region R and if 'a' is any point within c, then $f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz,$ the integration around C being taken in the positive direction .

2. Evaluate $\int_C \frac{e^{-z}}{z^2}dz$ **,where C is a circle** $|z|=1$**.** **(AU-2012)**

We know that $\int_c \frac{f(z)}{z-a}dz = 2\pi i f(a)$, $\int_c \frac{e^{-z}dz}{z^2} = \int_c \frac{e^{-z}}{(z-0)^2}$

Here $f(z) = e^{-z}$, a= 0 is lies inside $|z|=1$

By cauchy's integral formula we get

$$\int_C \frac{e^{-z}}{z^2}dz = 2\pi i f'(a) = 2\pi i(-1) = -2\pi i$$

3. **Evaluate** $\int_C \frac{z^2+1}{z^2-1}dz$ **where C is a circle of unit radius and centre at z=i.** **(AU-2013)**

$|z-i|=1$

The poles z=1,z=-1 lies outside the circle

$\therefore \frac{z^2+1}{z^2-1}$ is analytic inside $|z-i|=1$

By Cauchy's theorem, $\int_C \frac{z^2+1}{z^2-1}dz$ **=0**

4. Evaluate $\int_z \sec z dz$ **where c is the unit circle** $|z|=1$ **(AU-2014)**

$$\int_z \sec z dz \quad = \int_z \frac{1}{\cos z}dz$$

The pole are given by the solution of cos z = 0

$$i.e., z = (2n+1)\frac{\pi}{2}, n = 0,1,2,.......$$

$$z = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2,}.........$$

Hence all the poles lies outside $|z|=1$, secz is analytic with $|z|=1$

By Cauchy's theorem $\int_z \sec z dz = 0$

5. Evaluate $\int_C \frac{3z^2+7z+1}{(z+1)}dz$ **where C is |z|=1/2** **(AU-2013)**

$\int_C \frac{3z^2+7z+1}{z-(-1)}dz$ Here z=-1 lies outside c.

$\therefore f(z)$ is analytic inside and on c

$\therefore f'(z)$ is continuous inside c.

Hence by cauchy' s theorem $\int_c f(z)dz = 0$

6. State Taylor's theorem. **(AU-2011)**

A function f(z) , is analytic inside a circle C with centre at a ,can be expanded in the series

$$f(z)= f(a)+ f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\cdots\cdots+\frac{f^{(n)}(a)}{n!}(z-a)^n+\cdots\cdots to\ \infty$$

Which is convergent at every point inside C

7. Find the Taylor series of the function f(z)=sinz about z=π/4 **(AU-2013)**

$f(z)=\sin z$

$f'(z)=\cos z$

$f''(z)=-\sin z$

$f'''(z)=-\cos z$

Here $a=\frac{\pi}{4}$, $f(\frac{\pi}{4})=\sin(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f'(\frac{\pi}{4})=\cos(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$, $f''(\frac{\pi}{4})=-\sin(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f'''(\frac{\pi}{4})=-\cos(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

Taylor's series is $f(z)=f(\frac{\pi}{4})+\frac{z-\frac{\pi}{4}}{1!}f'(\frac{\pi}{4})+\frac{(z-\frac{\pi}{4})^2}{2!}f''(\frac{\pi}{4})+........$

8. Find the Laurent's series for the function f(z)=z$^2 e^{1/z}$ about z=0 **(AU-2013)**

$$z^2e\frac{1}{z}=z^2\left[1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}.........\right]$$

$$=z^2+z+\frac{1}{2}+..........$$

9. Define singular point. **(AU-2012)**

A point z=z_0 at which a function f(z)fails to be analytic is called a singular point or singularity of f(z).

10. Identify the types of singularities of the following function $f(z)=e^{\frac{1}{(z-1)}}$ **(AU-2009)**

Here z=1 is a singular point

At z=1 , we get $f(z)=e^{\frac{1}{0}}=\infty$ which is not defined.

Also z=1 is not a pole or removable singularity

Z=1 is an essential singularity.

11. Discuss the nature of the singularities of the function f(z)=$\frac{\sin z}{z}$ **(AU-2012)-2**

Poles of f(z) are obtained by equating the denominator to zero

i.e $f(z)=\frac{\sin z}{z}$

z=o is a pole of order 1

sinz=0

z=n π where n= 0, $n=0,\pm1,\pm2...$

12.Identify the type of singularity of function sin(1/1-z) **(AU-2015)**

Sin(1/1-z)=1/1-z-1/3!(1/1-z)^3+1/5!(1/1-z)^5

The RHS is the Laurent series with infinite number of terms about the singular part z=1,z=1 is an essential singularity of f(z).

13. Find the nature of the singularity z=0 of the function $f(z)=\dfrac{1-\cos z}{z^2}$ **(AU-2011)**

Poles of f(z) are obtained by equating the denominator to zero

i.e $f(z)=\dfrac{1-\cos z}{z^2}$

$\mathbf{z^2}=0$ is a pole of order 2

14. State Cauchy's residue theorem. **(AU-2014)**

If f(z) be an analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $z_1,z_2,z_3,\ldots..z_n$ inside c then

$\int f(z)dz = 2\pi i$ [sum of the residues of f(z) at $z_1,z_2,z_3,\ldots..z_n$] $=2\pi i\sum_{i=1}^{n} R_i$

where R_i is the residue of f(z) at z=zi

15. If f(z)= $\dfrac{-1}{z-1}-2\left[1+(z-1)+(z-1)^2+\cdots\cdot\right]$**,find the residue of f(z) at z=1** **(AU-2012)**

Residue of $f(z)$ at $z=1$ is -1 (the coefficient of $\dfrac{1}{z-1}$)

16. Find the residue of $\dfrac{1-e^{2z}}{z^4}$ **at z=0** **(AU-2013)**

Given $f(z)=\dfrac{1-e^{2z}}{z^4}$

Here z = 0 is a pole of order 4

$$\operatorname{Re}s(z=0)=\frac{1}{3!}\underset{z\to 0}{Lt}\frac{d^3}{dz^3}\left[(z-0)^4\frac{1-e^{2z}}{z^4}\right]$$

$$=\frac{1}{6}\underset{z\to 0}{Lt}\frac{d^3}{dz^3}\left[1-e^{2z}\right]=-\frac{4}{3}$$

17. Find the residue of the function f(z)= $\dfrac{4}{z^3(z-2)}$ **at a simple pole** **(AU-2012)**

$$f(z)=\frac{4}{z^3(z-2)}=\frac{4}{(z-0)^3(z-2)}$$

Here z = 0 is a pole of order 3 and z=2 is a pole of order 1

$$\operatorname{Re}s(z=0)=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d^2}{dz^2}\left[(z-0)^3\frac{4}{(z-0)^3(z-2)}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\frac{d^2}{dz^2}\left[\frac{4}{(z-2)}\right]=\frac{1}{2}\underset{z\to 0}{Lt}\left[\frac{8}{(z-2)^3}\right]=\frac{1}{2}$$

$$\operatorname{Re}s(z=2)=\underset{z\to 2}{Lt}\left[(z-2)\frac{4}{(z-0)^3(z-2)}\right]=\frac{1}{8}$$

18. Find the residue of $f(z)$ **=** $\dfrac{z+1}{(z-1)(z-2)}$ **at z=2** **(AU-2012)**

$$\operatorname{Re}s(z=2)=\underset{z\to 2}{Lt}\left[(z-2)\frac{z+1}{(z-1)(z-2)}\right]=3$$

19. Find the resuidue of cot z at the pole z=0. **(AU-2010)**

f(z)=cotz= $\dfrac{\cos z}{\sin z}$ Poles of f(z) are sinz = 0 = sin n π

$z = n\pi$, where n = 0,±1, ±2,….

$[\text{Resf}(z)]_{z=n\pi} = \lim_{z\to n\pi}(z-n\pi)\frac{\cos z}{\sin z} = \lim_{z\to n\pi}\frac{-(z-n\pi)\sin z+\cos z(1)}{\cos z}$ (by L' Hospital rule)

$[\text{Resf}(z)]_{z=n\pi} = 1$

20. Determine the residue of $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ **at z=1** **(AU-2012)**

Given $f(z)=\frac{z^2}{(z-1)^2(z+2)}$

Here z =1 is a pole of order 2

$$\text{Res}[z=z_0]=Lt_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[z-z_0]^m f(z)$$

$$\text{Res}[z=1]=lt_{z\to 1}\frac{d}{dz}\left[(z-1).^2\frac{z^2}{(z-1)^2(z+2)}\right]=\underset{z\to 1}{Lt}\frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

$$= f(z)=\frac{5}{9}$$

21. Find the residue of $f(z)=\frac{50z}{(z+4)(z-1)^2}$ **at z = 1** **(AU-2009)**

Z = 1 is a pole of order 2

$$\text{Res}[f(z)]_{z=1} = \underset{z\to 1}{Lt}\frac{d}{dz}\left[(z-1)^2\frac{50z}{(z+4)(z-1)^2}\right]$$

$$=\underset{z\to 1}{Lt}\left[\frac{(z+4)50-50z}{(z+4)^2}\right]=\frac{250-50}{25}=8$$

22. Evaluate $\int_c \frac{e^z}{z-1}\,dz$ **if c is |z|=2** **(AU-2010)**

Z=1 is a pole of order 1 which lies inside |z|=2

$$\int_c \frac{e^z}{z-1}\,dz = 2\pi i f(1)$$

$$=2\pi i e$$

PART-B

1\. a .Evaluate $\int_C \frac{z}{(z-1)(z-2)^2}dz$ here C is $|z-2|=\frac{1}{2}$ by using Cauchy's integral formula.

(AU-2012)(8)

b .Evaluate $\int \frac{7z-1}{z^2-3z-4}dz$ over the curve C:$x^2+4y^2=4$ using Cauchy's integral formula.

(AU-2013)(8)

2.a. Evaluate $\int_C \frac{z+1}{(z^2+2z+4)^2}dz$ where c is the circle $|z+1+i|=2$ by Cauchy's integral formula.

(AU-2013)(8)

b. Evaluate $\int_C \frac{z+4}{z^2+2z+5}dz$ where C is the circle |z+1+i|=2 using Cauchy's integral formula.

(AU-2013)(8)

3.a.Using Cauchy's integral formula ,evaluate $\int_C \frac{e^z}{(z+1)^2(z+2)}dz$ where C is |z|=3. **(8)**

b. If $f(a)=\int_C \frac{13z^2+27z+15}{z-a}dz$ where c is the circle $|z|=2$ then find

$f(3), f'(1-i), f''(1-i)$ *and* $f(1-i)$ **(AU-2014)(8)**

4.a. Evaluate $\int_C \frac{z^3}{(2z+i)^3}dz$ where c is the unit circle $|z|=1$ **(8)**

b. Obtain Taylor's series for $f(z)=\frac{2z^3}{z(z+1)^3}$ about z=i **(AU-2013)(8)**

5.a. Evaluate $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent series valid for the regions |z|>3 and 1<|z|<3

(AU-2013)(8)

b. Find the Laurent's series expansion of $f(z)=\frac{7z-2}{(z-2)(z+1)}$ valid in the region

$|z+1|<1$ and $|z+1|>3$ **(8)**

6.a. Expand the function $f(z)=\frac{z^2-1}{z^2+5z+6}$ in Laurent's series |z|>3 **(AU-2013)(8)**

b. Obtain the Laurent's series expansion of $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ in $2<|z|<3$ **(AU-2015)(8)**

7.a.Expand $f(z)= 1/ z^2-4z+3$ as the Laurent's series expansion of **1<**$|z|<3$ **(AU-2014)(8)**

b. Obtain the Laurent's series expansion of $f(z)= 1/_{z-z^2}$ in the region 1<|z+1|<2 and|z+1|>2.

(AU-2014)(8)

8.a. Evaluate $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz$ where C is |z|=3 Using Cauchy's Residue theorem **(AU-2013) (8)**

b. Using Cauchy's residue theorem evaluate $\int_C \frac{z-1}{(z-1)^2(z-2)}dz$

where C is $|z-i|=2$ **(AU-2014)(8)**

9.a. Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)}dz$ where C is |z|=3 **(AU-2015) (8)**

b. Evaluate $\int_0^{2\pi} \frac{dx}{(x^2+a^2)^2}$, a>0 using contour integration. **(AU-2015)(8)**

10.a.Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta$ using contour integration **(AU-2013)(8)**

b. Using contour integration on unit circle , evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$ **(AU-2014)(8)**

11**.**a. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **(AU-2014)(8)**

b. Using contour integration ,evaluate the integral $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta$ **(AU-2013)(8)**

12.a. Evaluate $\int_0^{\infty} \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$ $,a>0,b>0$ **(AU-2013)(8)**

b. Evaluate using contour integration $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$ **(AU-2014)(8)-2**

13.a. Using contour integration prove that $\int_0^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}=\frac{\pi}{6}$ **(AU-2013)(8)**

b. Using contour integration on unit circle , evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+9)}$ **(AU-2014)(8)**

14.a. Evaluate $\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2} dx$,using contour integration. **(AU-2012)(8)**

b. Show that $\int_{-\infty}^{\infty} \frac{x^2 - x + 2dx}{(x^4+10x^2+9)} dx = \frac{5\pi}{2}$ **(AU-2013)(8)**

15.a.S.T. $\int_{0}^{\infty} \frac{dx}{(1+x^4)} = \frac{\prod}{2\sqrt{2}}$ **(8)**

b.Evaluate $\int_{0}^{2\pi} \frac{d\theta}{1-2x\sin\theta+x^2}$ (0<x<1),using contour integration. **(8)**